

April 3, 2008

Fredholm equations for non-symmetric kernels, with applications to iterated integral operators.

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Abstract: We give the Jordan form and the Singular Value Decomposition for an integral operator \mathcal{N} with a non-symmetric kernel $N(y, z)$. This is used to give solutions of Fredholm equations for non-symmetric kernels, and to determine the behaviour of \mathcal{N}^n and $(\mathcal{N}\mathcal{N}^*)^n$ for large n .

1 Introduction and summary

Suppose that $\Omega \subset \mathbb{R}^p$ and that μ is a σ -finite measure on Ω . Consider a $s_1 \times s_2$ complex matrix function $N(y, z)$ on $\Omega \times \Omega$. Generally we shall assume that $N \in L_2(\mu \times \mu)$ and is non-trivial, that is,

$$0 < \|\mathcal{N}\|_2^2 = \int \int \|N(y, z)\|^2 d\mu(y) d\mu(z) < \infty.$$

The integral operator associated with (N, μ) is \mathcal{N} defined by

$$\mathcal{N}q(y) = \int N(y, z)q(z)d\mu(z), \quad p(z)^*\mathcal{N} = \int p(y)^*N(y, z)d\mu(y) \quad (1.1)$$

where $p : \Omega \rightarrow C^{s_1}$ and $q : \Omega \rightarrow C^{s_2}$ are any functions for which these integrals exist, for example $p, q \in L_2(\mu)$, that is $\int |p|^2 d\mu < \infty$, and similarly for q . (All integrals are over Ω . * denotes the transpose of the complex conjugate.)

Section 2 reviews Fredholm theory for Hermitian kernels, that is, when $N(y, z)^* = N(z, y)$, so that $s_1 = s_2$. For this case, $\mathcal{N}^n = O(r_1^n)$ for large n where r_1 is the magnitude of the largest eigenvalue.

Section 3 extends this to non-Hermitian kernels for the case of diagonal Jordan form. Again $\mathcal{N}^n = O(r_1^n)$ for large n for r_1 as before.

Section 4 deals with the case of non-diagonal Jordan form. In this case $\mathcal{N}^n = O(r_1^n n^{M-1})$ for large n for r_1 as before where M is the largest multiplicity of those eigenvalues with modulus r_1 .

Section 5 gives the Singular Value Decomposition (SVD) for a non-symmetric kernel $N(y, z)$. In this case one has results such as $(\mathcal{N}\mathcal{N}^*)^n = O(\theta_1^{2n})$ for large n where θ_1 is the largest singular value..

2 Hermitian kernels

Matrix theory

First consider a Hermitian matrix $N^* = N \in C^{s \times s}$. Its eigenvalues ν_1, \dots, ν_s are the roots of $\det(N - \nu I) = 0$. They are real. Corresponding to ν_j is an eigenvector p_j satisfying $Np_j = \nu_j p_j$. These are orthonormal: $p_j^* p_k = \delta_{jk}$ where $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \neq k$. Set $P = (p_1, \dots, p_s)$. If N and its eigenvalues are real, then P can be taken to be real. The *spectral decomposition* of N in terms of its eigenvalues and eigenvectors is

$$\begin{aligned} N &= P\Lambda P^* = \sum_{j=1}^s \nu_j p_j p_j^* \text{ where } \Lambda = \text{diag}(\nu_1, \dots, \nu_s), \\ \sum_{j=1}^s \nu_j p_j p_j^* &= PP^* = I_s = P^* P = (p_j^* p_k). \end{aligned} \quad (2.1)$$

So for $\alpha \in C$

$$N^\alpha = P\Lambda^\alpha P^* = \sum_{j=1}^s \nu_j^\alpha p_j p_j^*, \quad (2.2)$$

provided that if $\det(N) = 0$, then α has non-negative real part. So for $\det(N) \neq 0$,

$$Nf = g \Rightarrow f = N^{-1}g = \sum_{j=1}^s \nu_j^{-1} p_j (p_j^* g).$$

Similarly for ν not an eigenvalue and $f, g \in C^s$,

$$(\nu I - N)f = g \Rightarrow f = (\nu I - N)^{-1}g = \sum_{j=1}^s (\nu - \nu_j)^{-1} p_j (p_j^* g).$$

For large n , if

$$r_1 = |\nu_1| = \dots = |\nu_M| > r_0 = \max_{j=M+1}^s |\nu_j| \quad (2.3)$$

then

$$N^n = r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^M [\text{sign}(\nu_j)]^n p_j p_j^* = O(1) \quad (2.4)$$

assuming that s does not depend on n .

Function theory

Now consider a function $N(y, z) : \Omega^2 \rightarrow C^{s \times s}$ and a σ -finite measure μ on $\Omega \subset R^p$. Its integral operator with respect to μ is \mathcal{N} defined by (1.1). Suppose that the *kernel* N is Hermitian, that is,

$$N(y, z)^* = N(z, y).$$

Then the analogues of the matrix results above are as follows. Suppose that $\|\mathcal{N}\|_2^2 > 0$ and

$$\sum_{j=1}^s \int |N_{jj}(x, x) d\mu(x) < \infty.$$

The *spectral decomposition* of N in terms of its eigenvalues and vector eigenfunctions $\{p_j(y)\} : \Omega \rightarrow C^s$, is

$$N(y, z) = P(y) \Lambda P(z)^* = \sum_{j=1}^{\infty} \nu_j p_j(y) p_j(z)^* \quad (2.5)$$

where $\Lambda = \text{diag}(\nu_1, \nu_2, \dots)$, $P(y) = (p_1(y), p_2(y), \dots)$.

The eigenfunctions are orthonormal with respect to μ :

$$\int p_j^* p_k d\mu = \delta_{jk}.$$

In Fredholm theory the convention is to call $\{\lambda_j = \nu_j^{-1}\}$ the eigenvalues, rather than $\{\nu_j\}$.

$\mathcal{I}_{yz} = P(y)P(z)^*$ is generally divergent but can be thought of as a generalized Dirac function: $\int \mathcal{I}_{yz} f(z) d\mu(z) = f(y)$. $P(z)^* P(y) = (p_j(z)^* p_k(y))$ satisfies $\int P(z)^* P(z) d\mu(z) = I_{\infty}$.

For $n = 1, 2, \dots$, $N_n(y, z) = \mathcal{N}^{n-1} N(y, z)$ satisfies

$$N_n(y, z) = P(y) \Lambda^n P(z)^* = \sum_{j=1}^s \nu_j^n p_j(y) p_j(z)^*.$$

For large n , if (2.3) holds then

$$N_n(y, z) = r_1^n C_n(y, z) + O(r_0^n) \text{ where } C_n(y, z) = \sum_{j=1}^M [\text{sign}(\nu_j)]^n p_j(y) p_j(z)^* = O(1).$$

Conditions for (2.5) to hold pointwise and uniformly are given in Withers (1974, 1975, 1978.) It is known as Mercer's Theorem.

The resolvent

Given functions $f, g : \Omega \rightarrow C^s$, the *Fredholm integral equation of the second kind*,

$$p(y) - \lambda \mathcal{N} p(y) = f(y), \quad (2.6)$$

can be solved for λ not an eigenvalue using

$$(I - \lambda \mathcal{N})^{-1} = I + \lambda \mathcal{N}_{\lambda}, \text{ that is, } \mathcal{N}_{\lambda} = (I - \lambda \mathcal{N})^{-1} \mathcal{N}$$

where

$$\mathcal{N}_{\lambda} f(y) = \int N_{\lambda}(y, z) f(z) d\mu(z), \quad g(z) \mathcal{N}_{\lambda} = \int g(y) N_{\lambda}(y, z) d\mu(y),$$

and the *resolvent* of \mathcal{N} ,

$$N_{\lambda}(y, z) = (I - \lambda \mathcal{N})^{-1} N(y, z) : \mathcal{C} \times \Omega^2 \rightarrow \mathcal{C}^{s \times s}$$

with operator \mathcal{N}_λ is the unique solution of

$$(I - \lambda \mathcal{N})\mathcal{N}_\lambda = \mathcal{N} = \mathcal{N}_\lambda(I - \lambda \mathcal{N}),$$

that is,

$$\lambda \mathcal{N} N_\lambda(y, z) = N(y, z) - N_\lambda(y, z) = \lambda N_\lambda(y, z) \mathcal{N}.$$

If this can be solved analytically or numerically, then one has a solution of (2.6) without the need to compute the eigenvalues and eigenfunctions of \mathcal{N} .

The resolvent satisfies

$$N_\lambda(y, z) = \sum_{j=1}^{\infty} p_j(y) p_j(z)^* / (\lambda_j - \lambda). \quad (2.7)$$

Conditions for this to hold are given by Corollary 3 of Withers (1975). The Fredholm equation of the second kind, (2.6), has solution

$$p(y) = f(y) + \lambda \sum_{j=1}^{\infty} p_j(y) \int p_j^* f d\mu / (\lambda_j - \lambda).$$

The resolvent exists except for $\lambda = \lambda_j$, an eigenvalue. The eigenvalues of \mathcal{N} are the zeros of its *Fredholm determinant*

$$D(\lambda) = \prod_{j=1}^{\infty} (1 - \lambda/\lambda_j) = \exp\left\{-\int_0^\lambda d\lambda \int \text{trace } N_\lambda(x, x) d\mu(x)\right\}. \quad (2.8)$$

The *Fredholm integral equation of the first kind*

$$\lambda \mathcal{N} p(x) = p(x)$$

has a solution provided that λ is an eigenvalue. For ν an eigenvalue, its general solution $p(x)$ is a linear combination of the eigenfunctions $\{p_j(x)\}$ corresponding to $\lambda_j = \lambda$.

Example 2.1 Suppose that $Y, Z \in R$ and $\begin{pmatrix} Y \\ Z \end{pmatrix} \sim \mathcal{N}_2(0, V)$ where $V = \begin{pmatrix} I & r \\ r & I \end{pmatrix}$. So V is the correlation matrix for $\begin{pmatrix} Y \\ Z \end{pmatrix}$. For $j \in N_+$, $x \in R$ set $p_j(x) = H_j(x)/j!^{1/2}$ where $H_j(x)$ is the standard univariate Hermite polynomial. Then $\int p_j p_k \phi_I = \delta_{jk}$ and

$$\sum_{j=0}^{\infty} r^j p_j(y) p_j(z) = \phi_C(y, z) / \phi_I(y) \phi_I(z).$$

This is Mehler's expansion for the standard bivariate normal distribution. Pearson gave an integrated version and Kibble extended it to an expansion for $\phi_V(x)/\phi_I(x)$ for $x \in R^k$ and V a correlation matrix. See for example (45.52) p127 and p321-2 of Kotz, Balakrishnan and Johnson (2000).

3 Functions of two variables with diagonal Jordan form

Diagonal Jordan form for matrices

Consider $N \in C^{s \times s}$ with eigenvalues ν_1, \dots, ν_s , the roots of $\det(N - \nu I) = 0$. N is said to have *diagonal Jordan form* (DJF) if

$$N = P\Lambda Q^* \text{ where } PQ^* = I \text{ and } \Lambda = \text{diag}(\nu_1, \dots, \nu_s).$$

So

$$\begin{aligned} N &= \sum_{j=1}^s \nu_j p_j q_j^*, \quad q_j^* p_k = \delta_{jk} \text{ where } P = (p_1, \dots, p_s), \quad Q = (q_1, \dots, q_s), \\ \sum_{j=1}^s p_j q_j^* &= PQ^* = I_s = Q^* P = (q_j^* p_k). \end{aligned}$$

If N, Λ are real then P, Q can be taken as real. Also

$$NP = P\Lambda, \quad Np_j = \nu_j p_j, \quad N^*Q = Q\bar{\Lambda}, \quad N^*q_j = \bar{\nu}_j q_j, \quad (3.1)$$

and for any complex α ,

$$N^\alpha = P\Lambda^\alpha Q^* = \sum_{j=1}^s \nu_j^\alpha p_j q_j^*$$

provided that if $\det(N) = 0$, then α has non-negative real part. Suppose that

$$\nu_j = r_j e^{i\theta_j} \text{ and } r_1 = \dots = r_R > r_0 = \max_{j=R+1}^s r_j. \quad (3.2)$$

Then

$$N^n = r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^R e^{in\theta_j} p_j q_j^* = O(1) \quad (3.3)$$

Taking $\alpha = -1$ gives the inverse of N when this exists.

Diagonal Jordan form for functions

Now consider a function $N(y, z) : \Omega^2 \rightarrow C^{s \times s}$. When N has diagonal Jordan form (for example when its eigenvalues are all different), then *the Fredholm equations of the first kind*,

$$\lambda \mathcal{N}p(y) = p(y), \quad \bar{\lambda} \mathcal{N}^*q(z) = q(z),$$

or equivalently for $\nu = \lambda^{-1}$,

$$\mathcal{N}p(y) = \nu p(y), \quad \mathcal{N}^*q(z) = \bar{\nu} q(z),$$

also have only a countable number of solutions, say $\{\lambda_j = \nu_j^{-1}, p_j(y), q_j(z), j \geq 1\}$ up to *arbitrary* constant multipliers for $\{p_j(y), j \geq 1\}$, satisfying the *bi-orthogonal* conditions

$$\int q_j^* p_k d\mu = \delta_{jk}.$$

These are called the *eigenvalues* and *right and left eigenfunctions* of (N, μ) or \mathcal{N} . Also

$$N(y, z) = \sum_{j=1}^{\infty} \nu_j p_j(y) q_j(z)^* = P(y) \Lambda Q(z)^* \quad (3.4)$$

where $\Lambda = \text{diag}(\nu_1, \nu_2, \dots)$, $P(y) = (p_1(y), p_2(y), \dots)$, $Q(z) = (q_1(y), q_2(y), \dots)$.

with convergence in $L_2(\mu \times \mu)$, or pointwise and uniform under stronger conditions. If N is a real function and Λ is real, then P, Q can be taken as real functions

For $n \geq 1$,

$$N_n(y, z) = \mathcal{N}^{n-1} N(y, z) \quad (3.5)$$

satisfies

$$\begin{aligned} N_n(y, z) &= \sum_{j=1}^{\infty} \nu_j^n p_j(y) q_j(z)^* = P(y) \Lambda^n Q(z)^*, \\ \mathcal{N}^n p(y) &= \sum_{j=1}^{\infty} \nu_j^n p_j(y) \int q_j^* p d\mu. \end{aligned} \quad (3.6)$$

If (3.2) holds then

$$\begin{aligned} N_n(y, z) &= r_1^n C_n + O(r_0^n) \text{ where } C_n = \sum_{j=1}^R e^{in\theta_j} p_j(y) q_j(z)^* = O(1) \\ \mathcal{N}^n p(y) &= r_1^n c_n + O(r_0^n) \text{ where } c_n = \sum_{j=1}^R e^{in\theta_j} p_j(y) \int q_j^* p d\mu = O(1). \end{aligned} \quad (3.7)$$

The resolvent satisfies the equations of Section 2 except that (2.7) is replaced by

$$N_\lambda(y, z) = \sum_{j=1}^{\infty} p_j(y) q_j(z)^* / (\lambda_j - \lambda). \quad (3.8)$$

The *Fredholm determinant* is again given by (2.8). If only a finite number of eigenvalues are non-zero, the *kernel* $N(y, z)$ is said to be *degenerate*. (For example this holds if μ puts weight only at n points.) If $R = 1$, that is,

$$|\lambda_1| < |\lambda_j| \text{ for } j > 1, \quad (3.9)$$

then as $n \rightarrow \infty$,

$$\mathcal{N}^{n+1} f(y) / \mathcal{N}^n f(y) \rightarrow \lambda_1^{-1}, \quad f(y) \mathcal{N}^{n+1} / f(y) \mathcal{N}^n \rightarrow \lambda_1^{-1}.$$

This is one way to obtain the first eigenvalue λ_1 arbitrarily closely. Another is to use

$$\lambda_1^{-1} = \sup \left\{ \int g \mathcal{N} h d\mu : \int g h d\mu = 1 \right\} \text{ if } \lambda_1 > 0, \quad (3.10)$$

$$\lambda_1^{-1} = \inf \left\{ \int g \mathcal{N} h d\mu : \int g h d\mu = 1 \right\} \text{ if } \lambda_1 < 0. \quad (3.11)$$

The maximising/minimising functions are the first eigenfunctions $g = g_1, h = h_1$. These are unique up to a constant multiplier if (3.9) holds. If λ_1 is known, one can use

$$(\lambda_1 \mathcal{N})^n f(y) \rightarrow p_1(y) \int q_1^* f d\mu, \quad f(y)^* (\lambda_1 \mathcal{N})^n \rightarrow q_1(y) \int f^* p_1 d\mu,$$

for any function $f : \Omega \rightarrow \mathcal{C}^s$, to approximate $p_1(y), q_1(y)$. One may now repeat the procedure on the operator \mathcal{N}_1 corresponding to

$$N_1(y, z) = N(y, z) - \nu_1 p_1(y) q_1(z)^*$$

to approximate $\lambda_2, p_2(y), q_2(z)$ assuming that the next eigenvalue in magnitude, λ_2 , has multiplicity 1.

For further details see Withers (1974, 1975, 1978) and references.

4 Fredholm theory for non-diagonal Jordan form

Non-diagonal Jordan form for matrices

For $N \neq N^*$ a matrix in $C^{s \times s}$, its general Jordan form is

$$N = PJP^{-1} \text{ where } J = \text{diag}(J_1, \dots, J_r), \quad J_j = J_{m_j}(\lambda_j), \quad \sum_{j=1}^r m_j = s,$$

$$J_m(\lambda) = \lambda I_m + U_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad (4.1)$$

for some matrix P , and U_m is the $m \times m$ matrix with 1s on the superdiagonal and 0s elsewhere:

$$(U_m)_{jk} = \delta_{j,k-1}.$$

(See [1] for example. If N and its eigenvalues are real, then P can be taken as real.) So for $n \geq 1$,

$$N^n = PJ^nP^{-1} \text{ where } J^n = \text{diag}(J_1^n, \dots, J_r^n).$$

By the Binomial Theorem,

$$J_m(\lambda)^n = \sum_{a=0}^n \binom{n}{a} \lambda^{n-a} U_m^a \text{ and } (U_m^a)_{jk} = \delta_{j,k-a}.$$

So $U_m^m = 0$. For example

$$J_2(\lambda)^n = \lambda^n I_2 + n\lambda^{n-1} U_2 = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

So N^n can be expanded in block matrix form

$$(N^n)_{jk} = \sum_{c=1}^r P_{jc} J_c^n P^{ck}$$

where we partition P and its inverse as

$$P = (P_{jk} : j, k = 1, \dots, r), \quad P^{-1} = (P^{jk} : j, k = 1, \dots, r)$$

with elements P_{jk} and P^{jk} matrices in $C^{m_j \times m_k}$.

Alternatively setting

$$Q^* = P^{-1}, \quad (P_1, \dots, P_r) = P, \quad (Q_1, \dots, Q_r) = Q,$$

with $P_j, Q_j \in C^{s \times m_j}$, we have

$$\begin{aligned} N^n &= PJ^n Q^* = \sum_{j=1}^r P_j J_j^n Q_j^*, \\ \sum_{j=1}^r P_j Q_j^* &= PQ^* = I_s = Q^* P = (Q_j^* P_k) \end{aligned} \quad (4.2)$$

so that

$$Q_j^* P_j = I_{m_j}, \quad Q_j^* P_k = 0 \in C^{m_j \times m_k} \text{ if } j \neq k.$$

P can be obtained as follows. Let p_{jk} be the k th column of P_j for $k = 1, \dots, m_j$. Then

$$NP = PJ \Rightarrow NP_j = P_j J_j \Rightarrow Np_{jk} = \lambda_j p_{jk} + p_{j,k-1} \text{ where } p_{j0} = 0. \quad (4.3)$$

So one first obtains p_{j1} , the right eigenvector of N , then p_{j2}, \dots, p_{jm_j} . This is called *the Jordan chain*. Q can either be obtained by inverting P or using

$$N^* Q = QJ^* \Rightarrow N^* Q_j = Q_j J_j^* \Rightarrow N^* q_{jk} = \bar{\lambda}_j q_{jk} + q_{j,k+1} \text{ where } q_{j,m_j+1} = 0. \quad (4.4)$$

So one first computes q_{j,m_j} , the right eigenvector of N^* then $q_{j,m_j-1}, \dots, q_{j1}$. For large n and $\lambda \neq 0$,

$$J_n(\lambda)^n = \binom{n}{m-1} \lambda^{n-m+1} [U_m^{m-1} + O(1)]$$

and U_m^{m-1} is a matrix of 0's except for a 1 in its upper right corner. So if (3.2) holds and

$$m_1 = \dots = m_M > \max_{j=M+1}^R m_j,$$

then

$$\begin{aligned} (N^n)_{jk} &= \binom{n}{M-1} r_1^{n-M+1} [D_n + O(n^{-1})] \\ \text{where } D_n &= \sum_{c=1}^M P_{jc} e^{i(n-M+1)\theta_c} U_M^{M-1} P^{ck} = \sum_{c=1}^M P_{jc} e^{i(n-M+1)\theta_c} U_M^{M-1} P^{ck} = O(1). \end{aligned}$$

See Withers and Nadarajah (2008) for more details.

Non-diagonal Jordan form for functions

Now consider $N : \Omega^2 \rightarrow C^{s \times s}$. Suppose that μ is a σ -finite measure on Ω and that N is not Hermitian, that is $N(y, z)^* \neq N(z, y)$. Its Jordan form is

$$N(y, z) = P(y)JP(z)^{-1} = \text{ where } J = \text{diag}(J_1, J_2, \dots), \quad J_j = J_{m_j}(\lambda_j) \quad (4.5)$$

for $P(y) : \Omega \rightarrow C^{s \times \infty}$ and $J_m(\lambda)$ of (4.1) above. So partitioning

$$P(y) = (P_{jk}(y) : j, k = 1, 2, \dots), \quad P(z)^{-1} = (P^{jk}(z) : j, k = 1, 2, \dots),$$

with elements $P_{jk}(y)$ and $P^{jk}(z)$ matrix functions in $\Omega \rightarrow C^{m_j \times m_k}$, we can partition the n th iterated kernel, $N_n(y, z) = \mathcal{N}^{n-1}N(y, z)$ as

$$[N_n(y, z)]_{jk} = \sum_{c=1}^{\infty} P_{jc}(y) J_c^n P^{ck}(z).$$

Alternatively setting

$$Q(z)^* = P(z)^{-1}, \quad (P_1(y), P_2(y), \dots) = P(y), \quad (Q_1(z), Q_2(z), \dots) = Q(z),$$

with $P_j(y), Q_j(z) : \Omega \rightarrow C^{s \times m_j}$, we have

$$N_n(y, z) = P(y) J^n Q(z)^* = \sum_{j=1}^{\infty} P_j(y) J_j^n Q_j(z)^*. \quad (4.6)$$

$P(y)$ can be obtained as follows. Let $p_{jk}(y)$ be the k th column of $P_j(y)$ for $k = 1, \dots, m_j$. Then

$$\mathcal{N}P(y) = P(y)J \Rightarrow \mathcal{N}P_j(y) = P_j(y)J_j \Rightarrow \mathcal{N}p_{jk}(y) = \lambda_j p_{jk}(y) + p_{j,k-1}(y) \quad (4.7)$$

where $p_{j0}(y) = 0$. So one first obtains $p_{j1}(y)$, the right eigenfunction of N , then $p_{j2}(y), \dots, p_{jm_j}(y)$. $Q(z)$ can either be obtained by inverting $P(z)$ or using

$$\mathcal{N}^*Q(z) = Q(z)J^* \Rightarrow \mathcal{N}^*Q_j(z) = Q_j(z)J_j^* \Rightarrow \mathcal{N}^*q_{jk}(z) = \bar{\lambda}_j q_{jk}(z) + q_{j,k+1}(z) \quad (4.8)$$

where $q_{j,m_j+1}(z) = 0$. So one first computes $q_{j,m_j}(z)$, the right eigenfunction of N^* then $q_{j,m_j-1}(z), \dots, q_{j1}(z)$.

So if (3.2) holds and

$$m_1 = \dots = m_M > \max_{j=M+1}^R m_j,$$

then

$$(N_n(y, z))_{jk} = \binom{n}{M-1} r_1^{n-M+1} (D_{jkn}(y, z) + O(n^{-1}))$$

$$\text{where } D_{jkn}(y, z) = \sum_{c=1}^M e^{i(n-M+1)\theta_c} P_{jc}(y) U_M^{M-1} P^{ck}(z) = O(1)$$

has (a, b) element $\sum_{c=1}^M e^{i(n-M+1)\theta_c} [P_{jc}(y)]_{a1} [P^{ck}(z)]_{Mb}$.

Example 4.1

5 The SVD for functions of two variables

The SVD for matrices

Suppose that $N \in C^{s_1 \times s_2}$. That is, N is a $s_1 \times s_2$ complex matrix. Denote its complex conjugate transpose by N^* . Its SVD is

$$N = PDQ^* = \sum_{j=1}^r \theta_j p_j q_j^* \text{ where } PP^* = I, \quad QQ^* = I, \quad r = \min(s_1, s_2), \quad (5.1)$$

$$P = (p_1, \dots, p_{s_1}) \in C^{s_1 \times s_1}, \quad Q = (q_1, \dots, q_{s_2}) \in C^{s_2 \times s_2}, \quad \theta_1 \geq \dots \geq \theta_r > 0$$

and for $s_1 = s_2$, $s_1 > s_2$, $s_1 < s_2$

$$D = \Lambda, \begin{pmatrix} \Lambda \\ 0 \end{pmatrix}, (\Lambda, 0) \text{ respectively where } \Lambda = \text{diag}(\theta_1, \dots, \theta_r).$$

If N is real, then so are P and Q .

So for $s_1 > s_2$,

$$DD^* = \begin{pmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{pmatrix}, D^*D = \Lambda^2$$

and for $s_1 < s_2$,

$$DD^* = \Lambda^2, D^*D = \begin{pmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Compare this with (3.1). Also for $1 \leq j \leq r$,

$$Nq_j = \theta_j p_j, N^*p_j = \theta_j q_j,$$

for $r < j \leq s_1$, $Nq_j = 0$, and for $r < j \leq s_2$, $N^*p_j = 0$. Also since

$$NN^*P = PDD^*, N^*NQ = QD^*D,$$

the p_j is a right eigenvector of NN^* with eigenvalue θ_j^2 (or 0 if $r < j \leq s_1$) and the q_j is a right eigenvector of N^*N with eigenvalue θ_j^2 (or 0 if $r < j \leq s_2$). So (or by Section 2),

$$\begin{aligned} (NN^*)^n &= \sum_{j=1}^r \theta_j^{2n} p_j p_j^*, (N^*N)^n = \sum_{j=1}^r \theta_j^{2n} q_j q_j^* \text{ for } n \geq 1, \\ (NN^*)^n N &= \sum_{j=1}^r \theta_j^{2n+1} p_j q_j^*, (N^*N)^n N^* = \sum_{j=1}^r \theta_j^{2n+1} q_j p_j^* \text{ for } n \geq 0. \end{aligned} \quad (5.2)$$

These do not depend on the vectors $\{p_j, q_j, j \geq r\}$.

So if $\theta_1 = \dots = \theta_M > \theta_{M+1}$, then we have the approximations for $n \geq 0$,

$$\begin{aligned} (NN^*)^n &= \theta_1^{2n} \sum_{j=1}^M p_j p_j^* + O(\theta_{M+1}^{2n}), (N^*N)^n = \theta_1^{2n} \sum_{j=1}^M q_j q_j^* + O(\theta_{M+1}^{2n}) \text{ for } n \geq 1, \\ (NN^*)^n N &= \theta_1^{2n+1} \sum_{j=1}^r p_j q_j^* + O(\theta_{M+1}^{2n+1}), (N^*N)^n N^* = \theta_1^{2n+1} \sum_{j=1}^r q_j p_j^* + O(\theta_{M+1}^{2n+1}). \end{aligned} \quad (5.3)$$

But

$$I_{s_1} = (NN^*)^0 = \sum_{j=1}^{s_1} p_j p_j^*, I_{s_1} = (N^*N)^0 = \sum_{j=1}^{s_2} q_j q_j^*.$$

If $s_1 = s_2$ and N is non-singular, its inverse is

$$N^{-1} = Q\Lambda^{-1}P^*.$$

However unlike Jordan form, the SVD does not give a nice form for powers of N .

Now suppose $\Omega \subset R^p$ and that μ is a σ -finite measure on Ω . Consider a function $N(y, z) : \Omega^2 \rightarrow C^{s_1 \times s_2}$.

The equations

$$\mathcal{N}q(y) = \theta p(y), \quad \mathcal{N}^*p(z) = \theta q(z),$$

have a countable number of solutions, say $\{\theta_j, p_j(y), q_j(z), j \geq 1\}$ satisfying

$$\int p_j^* p_k d\mu = \int q_j^* q_k d\mu = \delta_{jk}.$$

The *singular values* $\{\theta_j\}$ may be taken as real, non-negative and non-increasing. (For convenience we have included $\theta_j = 0$.) $\{p_j(y)\}$ and $\{q_j(z)\}$ are the right eigenfunctions of $\mathcal{N}\mathcal{N}^*$ and $\mathcal{N}^*\mathcal{N}$ respectively, with eigenvalues $\{\theta_j^2\}$. Also in $L_2(\mu \times \mu)$

$$N(y, z) = \sum_{j=1}^{\infty} \theta_j p_j(y) q_j(z)^*. \quad (5.4)$$

If N is real, then so are $\{p_j, q_j\}$. By (5.4), for $n \geq 0$,

$$\begin{aligned} (\mathcal{N}\mathcal{N}^*)^n N(y, z) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} p_j(y) q_j(z)^*, \\ (\mathcal{N}^*\mathcal{N})^n N(y, z)^* &= \sum_{j=1}^{\infty} \theta_j^{2n+1} q_j(z) p_j(y)^*, \\ \mathcal{N}^*(\mathcal{N}\mathcal{N}^*)^n N(y, z) &= \sum_{j=1}^{\infty} \theta_j^{2n+2} q_j(y) q_j(z)^*, \\ \mathcal{N}(\mathcal{N}^*\mathcal{N})^n N(y, z)^* &= \sum_{j=1}^{\infty} \theta_j^{2n+2} p_j(z) p_j(y)^*, \end{aligned}$$

$$\begin{aligned} (\mathcal{N}\mathcal{N}^*)^n p(y) &= \sum_{j=1}^{\infty} \theta_j^{2n} p_j(y) \int p_j^* p d\mu, \\ (\mathcal{N}^*\mathcal{N})^n q(z) &= \sum_{j=1}^{\infty} \theta_j^{2n} q_j(z) \int q_j^* q d\mu, \\ (\mathcal{N}\mathcal{N}^*)^n \mathcal{N}q(y) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} p_j(y) \int q_j^* q d\mu, \\ (\mathcal{N}^*\mathcal{N})^n \mathcal{N}^*p(y) &= \sum_{j=1}^{\infty} \theta_j^{2n+1} q_j(y) \int p_j^* p d\mu, \end{aligned}$$

$$\int \text{trace } \mathcal{N}^*(\mathcal{N}\mathcal{N}^*)^n N(y, z)|_{z=y} d\mu(y) = \int \text{trace } \mathcal{N}(\mathcal{N}^*\mathcal{N})^n N(y, z)|_{z=y} d\mu(y) = \sum_{j=1}^{\infty} \theta_j^{2n+2}.$$

So if $\theta_1 = \dots = \theta_M > \theta_{M+1}$, then we have approximations such as

$$(\mathcal{N}\mathcal{N}^*)^n p(y) = \theta_1^{2n} \sum_{j=1}^{\infty} p_j(y) \int p_j^* p d\mu + O(\theta_{M+1}^{2n})$$

if $p_j(y) \int p_j^* p d\mu = O(1)$ for $j > M$. So for iterations of $\mathcal{N}^*\mathcal{N}$ or $\mathcal{N}\mathcal{N}^*$ the most important parameter is the largest singular value.

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